Some aspects of nonmeasurability with respct to selected σ-ideals

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$I \subset P(\mathbb{R})$ σ-ideal s.t.

1. I contain singletons, i.e. $[\mathbb{R}]^\omega \subseteq I$,
2. I has Borel base, i.e. $(\forall I \in I)(\exists B \in \mathrm{Borel} \cap I)(I \subseteq B)$,
3. I is translation invariant, i.e.

$$(\forall I \in I)(\forall x \in \mathbb{R})(x + I = \{x + i : \ i \in I\} \in I).$$

$\mathscr{N}$ σ-ideal of null sets and
$\mathscr{M}$ σ-ideal of all meager subsets of $\mathbb{R}$.

Definition

Let $A \subseteq \mathbb{R}$. We say that

1. A is I-nonmeasurable if A does not belong to the σ-algebra generated by Borel sets and σ-ideal I;
2. A is complete I-nonmeasurable if $A \cap B$ is I-nonmeasurable for every Borel set B which does not belong to I.

The folowing conditions are all equivalent:

1. A is completely I-nonmeasurable,
2. $A \cap B$ and $A \cap (\mathbb{R} \setminus B)$ does not belong to I for every Borel set B such that $B, \mathbb{R} \setminus B \notin I$,
3. A intersects every Borel set which does not belong to I and does not contan any of such sets.

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Steinhaus property

Definition
We say that I has Steinhaus property if

$$(\forall A \in \text{Borel} \setminus I)(\forall B \notin I)(A - B \text{ contains an open interval })$$

where $A - B = \{a - b : \ a \in A, b \in B\}$.
$\mathscr{M}$ and $\mathscr{N}$ has Steinhaus property but $\mathbb{R}^{\leq\omega}$ not.

Theorem

Assume I has Steinhaus property. Then there exists a partition $\mathcal{P} \subseteq I$ of $\mathbb{R}$ such that for every $\mathcal{A} \subseteq \mathcal{P}$

$$\bigcup \mathcal{A} \text{ is } I\text{-nonmeasurable}$$

$$\Downarrow$$

$$\bigcup \mathcal{A} \text{ is completely } I\text{-nonmeasurable.}$$

Finite sets

Theorem

Let $\mathcal{P} \subseteq [\mathbb{R}]^{<\omega}$ be a partition of $\mathbb{R}$. Then

1. *there is $\mathcal{A}_0 \subseteq \mathcal{P}$ such that $\bigcup \mathcal{A}_0$ is completely I-nonmeasurable;*
2. *there is $\mathcal{A}_1 \subseteq \mathcal{P}$ such that $\bigcup \mathcal{A}_1$ is I-nonmeasurable but is not completely I-nonmeasurable.*

Countable sets

Theorem

Assume that $\mathcal{P} \subseteq [\mathbb{R}]^{\leq\omega}$ is a point-countable cover of $\mathbb{R}$. Then we can find $\mathcal{A} \subseteq \mathcal{P}$ such that $\bigcup \mathcal{A}$ is completely $[\mathbb{R}]^{\leq\omega}$-nonmeasurable.

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There is $\mathcal{P} \subseteq [\mathbb{R}]^{\leq\omega}$ a partition of $\mathbb{R}$ such that for any $\mathcal{A} \subseteq \mathcal{P}$

$$\bigcup \mathcal{A} \text{ is } [\mathbb{R}]^{\leq\omega}\text{-nonmeasurable}$$

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$$\bigcup \mathcal{A} \text{ is completely } [\mathbb{R}]^{\leq\omega}\text{-nonmeasurable.}$$

Proof.

Enumerate $\{B_\xi : \xi < \omega_1\}$ of all perfect subsets of $\mathbb{R}$.
Choose partition $\mathcal{P} = \{X_\xi : \xi < \omega_1\} \subseteq \mathbb{R}^{\leq\omega}$ of $\mathbb{R}$ such that for every $\beta < \alpha$

$$B_\beta \cap X_\alpha \neq \emptyset$$

Set $\mathcal{A} \subseteq \mathcal{P}$ such that $|\mathcal{A}| = |\mathcal{P} \setminus \mathcal{A}| = \omega_1$
Then $B_\xi \cap \bigcup \mathcal{A} \neq \emptyset$ and $B_\xi \cap \bigcup(\mathcal{P} \setminus \mathcal{A}) \neq \emptyset$ $\square$

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Characterisation of Continuum Hypothesis

The following statements are equivalent:

1. CH,
2. there is $\mathcal{P} \subseteq [\mathbb{R}]^{\leq\omega}$ a partition of $\mathbb{R}$ such that for any $\mathcal{A} \subseteq \mathcal{P}$

$$\bigcup \mathcal{A} \text{ is } [\mathbb{R}]^{\leq\omega}\text{-nonmeasurable}$$

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$$\bigcup \mathcal{A} \text{ is completely } [\mathbb{R}]^{\leq\omega}\text{-nonmeasurable.}$$

Definition (Marczewski ideal s_0)

Let X be any fixed uncountable Polish space. Then we say that $A \in \mathcal{P}(X)$ is in s_0 iff

$$(\forall P \in Perf(X))(\exists Q \in Perf(X))\ Q \subseteq P \wedge Q \cap A = \emptyset.$$

s - measurability

Definition (s-measurable set)

Let X be any fixed uncountable Polish space. Then we say that $A \in \mathcal{P}(X)$ is s-**measurable** iff

$$(\forall P \in \mathrm{Perf}(X))(\exists Q \in \mathrm{Perf}(X))\ Q \subseteq P \wedge (Q \subseteq A \vee Q \cap A = \emptyset).$$

For every tree $T \subseteq \omega^{<\omega}$ let $[T]$ be an envelope of T which is defined as follows:

$$[T] = \{x \in \omega^\omega :\ (\forall n \in \omega)\ x \restriction n \in T\}.$$

I - measurability

A tree $T \subseteq \omega^{<\omega}$ is called a **Laver tree**, iff,

- $(\exists s \in T)(\forall t \in T) s \subseteq t \vee t \subseteq s$
- $(\forall t \in T) s \subseteq t \rightarrow \{n \in \omega : t^\frown n \in T\} \in [\omega]^\omega$

Definition (ideal I_0)

We say that $A \in \mathcal{P}(\omega^\omega)$ is in I_0 iff

$$(\forall T \in \mathrm{LT})(\exists Q \in \mathrm{LT})\ Q \subseteq T \wedge [Q] \cap A = \emptyset.$$

Definition (I-measurable set)

We say that $A \in \mathcal{P}(\omega^\omega)$ is I-**measurable** iff for every Laver tree $T \in \mathrm{LT}$ there is a Laver tree $S \in \mathrm{LT}$ such that

$$(S \subseteq T \wedge [S] \subseteq A) \vee (S \subseteq T \wedge [S] \cap A = \emptyset).$$

Miller tree

A tree $T \subseteq \omega^{<\omega}$ is called a **Miller tree**, iff,

- $(\exists s \in T)(\forall t \in T) s \subseteq t \vee t \subseteq s$
- $(\forall t \in T) s \subseteq t \to (\exists t' \in T)(t \subseteq t') \wedge \{n \in \omega : t^\frown n \in T\} \in [\omega]^\omega$

m_0 and m-measurability is defined as in Laver tree case.

Theorem

There exists a **m.a.d.** *family of functions* $\mathcal{A} \subseteq \omega^\omega$ *such that* $\mathcal{A}$ *is not* s, l, m*-measurable at the same time, and there is an dominating subfamily* $\mathcal{A}' \in [\mathcal{A}]^{\leq \mathfrak{d}}$ *in Baire space* ω^ω*.*

Proof

Let us inscribe dominating family $\mathscr{D} \in [\omega^\omega]^{\mathfrak{d}}$ into envelope of a.d. Laver tree $T \subseteq 4\mathbb{N}^{<\omega}$ and choose

- a.d. perfect tree $S \subseteq (4\mathbb{N}+1)^{<\omega}$
- a.d. Miller tree $M \subseteq (4\mathbb{N}+2)^{<\omega}$
- a.d Laver tree $L \subseteq (4\mathbb{N}+3)^{<\omega}$

Let us enumerate $\mathit{Perf}(S) = \{T_\alpha : \alpha < \mathfrak{c}\}$ a family of all perfect subsets of S and analogously $\mathit{Miller}(M) = \{M_\alpha : \alpha < \mathfrak{c}\}$, $\mathit{Laver}(L) = \{L_\alpha : \alpha < \mathfrak{c}\}$.

...

By transfinite recursion let us define

$$\{w_\alpha \in [S]^2 \times \omega^\omega \times [M]^2 \times \omega^\omega \times [L]^2 \times \omega^\omega : \alpha < \mathfrak{c}\}$$

where $w_\alpha = (a^s_\alpha, d^s_\alpha, x^s_\alpha, a^m_\alpha, d^m_\alpha, x^m_\alpha, a^l_\alpha, d^l_\alpha, x^l_\alpha)$ for any $\alpha < \mathfrak{c}$, and such that for any $\alpha < \mathfrak{c}$ we have:

1. $a^s_\alpha, d^s_\alpha \in S_\alpha$,
2. $\{a^s_\xi : \xi < \alpha\} \cap \{d^s_\xi : \xi < \alpha\} = \emptyset$,
3. $\{a^s_\xi : \xi < \alpha\} \cup \{x^s_\xi : \xi < \alpha\}$ is a.d.,
4. $\forall^\infty n\; x^s_\alpha(n) = d^s_\alpha(n)$ but $x^s_\alpha \neq d^s_\alpha$.
5. $a^m_\alpha, d^m_\alpha \in M_\alpha$,
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9. $a^l_\alpha, d^l_\alpha \in L_\alpha$,
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By transfinite induction theorem we have required sequence of the length $\mathfrak{c}$. Now set

$$A_s = \mathscr{D} \cup \{a_\alpha^s : \alpha < \mathfrak{c}\} \cup \{x_\alpha^s : \alpha < \mathfrak{c}\},$$

$$A_m = \mathscr{D} \cup \{a_\alpha^m : \alpha < \mathfrak{c}\} \cup \{x_\alpha^m : \alpha < \mathfrak{c}\}$$

and

$$A_l = \mathscr{D} \cup \{a_\alpha^l : \alpha < \mathfrak{c}\} \cup \{x_\alpha^l : \alpha < \mathfrak{c}\}$$

and let us extend the family $\mathcal{A} = \mathscr{D} \cup \mathcal{A}_s \cup \mathcal{A}_m \cup \mathcal{A}_l$ to any maximal a.d. family A. It is easy to check that A is required s, m and l-nonmeasurable m.a.d. family in the Baire space ω^ω with a dominating subfamily of size $\mathscr{D}$, what completes this proof.

Theorem

There are subsets A, B, C of the ω^ω such that

- *A is l-measurable and not s-measurable,*
- *B is m-measurable but not s-measurable,*
- *C is l-measurable but not m-measurable.*

Moreover, if $\mathfrak{b} = \mathfrak{c}$ then

- *there is a not l-measurable set which is s-measurable*
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Thank You for your attention